A short elementary proof of the Mohr-Mascheroni Theorem

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1 Introduction

In 1797 Lorenzo Mascheroni surprised the mathematical world with the theorem that every geometric construction that can be carried out by compasses and ruler may be done without ruler (see [1]). It turned out later that Georg Mohr proved this theorem in 1672 already (see [2]). The proofs given by Mohr and Mascheroni are quite complicated. Later easier proofs have been developed (see [3] or [4]). Furthermore the proof could be simplified by means of the circular inversion (see [5] or [6]). Here we give a very short and direct proof for the theorem that does not appeal to inversion.

2 The Mohr-Mascheroni Theorem

Theorem *Every geometric construction carried out by compasses and ruler can be done without ruler.*

Proof
We have to prove that the following three fundamental constructions are possible to carry out with compasses alone.

1. Points of intersection of two circles given by its centers and radii.
2. Points of intersection of a circle (given by center and radius) and a straight line (given by two points).
3. Point of intersection of two straight lines each of them given by two points.

There is nothing to prove for the intersection of two circles, so let us consider

2.1 Points of intersection of a circle and a straight line

Here we have to distinguish two cases:

1. The straight line misses the center of the circle.
2. The straight line passes through the center of the circle.

The first case is covered by the following construction:

Construction 1

If the straight line g is given by the points P_1 and P_2, we reflect the center M of the given circle K with respect to g as Figure 1 indicates. Then we find the two points of intersection $\{X, Y\} = K \cap g$ as the points of intersection of K and the reflected circle K'.

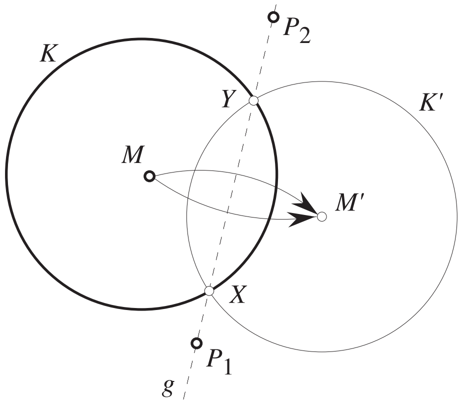

Figure 1

Before we are able to attack the second case, we need to have a construction which allows to bisect a segment AB without ruler. This can be done as follows:

Construction 2

Let K_1 be the circle through B with center A and K_2 the circle through A with center B with $K_1 \cap K_2 = \{C, D\}$ (see Figure 2). We then find a point E as the intersection of K_2 and the circle K_3 through D around C. Note that B is the bisection point of AE. Let F and G be the points of intersection of K_1 and the circle K_4 through A around E. Then we get the bisection point M of AB as intersection of the circles K_5 through A around F and K_6 through A around G.

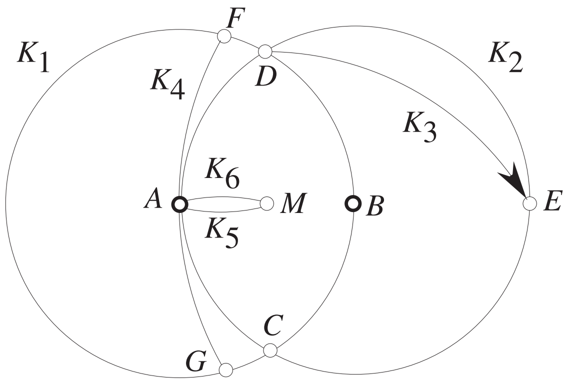

Figure 2

The correctness of the construction is evident: Note that the triangles FAM and EFA are similar with proportion $1 : 2$.

Remark 1 Note that AE has double the length of the segment AB!

Now we construct the points of intersection X and Y of a circle K with center M and a straight line MP:

Construction 3

Let A be an arbitrary point on K and $K \cap AP = \{A, B\}$ (see Figure 3). B is constructed according to construction 1. Let K_1 be a circle though A and B with radius lager than the radius R of K and M_1 the center of K_1. Now we construct a segment CD with endpoints on K_1 and length $2R$ (see Remark 1). Then we obtain P' as the intersection of CD and the circle K_2 through P around M_1 according to construction 1. Let M_3 be the bisection point of CD (see construction 2) and K_3 the circle around M_3 through C. Let E be a point of K_3 with $P'E = PB$. Now X and Y lie on K and $BX = EC$ and $BY = ED$.

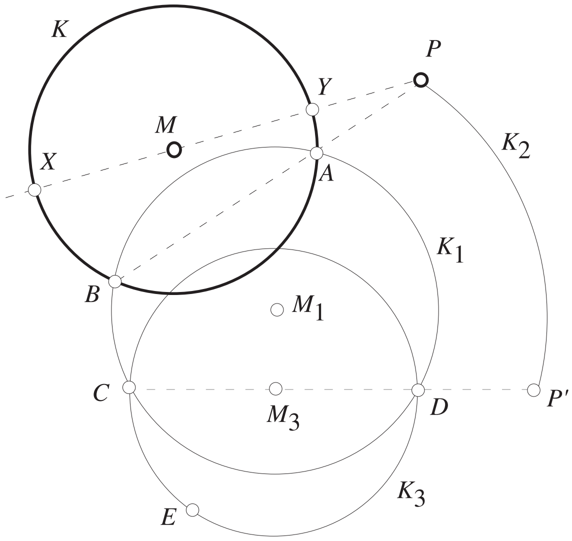

Figure 3

The correctness of the construction can bee verified as follows: Note that $PX \cdot PY = PA \cdot PB = P'C \cdot P'D$ by applying Euler's Theorem on intersecting secants, once for K and then for K_1. Hence the sets of points P, Y, M, X, B and P', D, M_3, C, E are congruent by construction. Thus in fact X and Y are obtained as described.

2.2 Point of intersection of two straight lines

Here we first need the following construction with compasses alone which allows to construct the footpoint L of the perpendicular through a point Q on a straight line P_1P_2:

Construction 4

Just reflect Q with respect to P_1P_2 (see Figure 4). If Q' is the reflected point, we find L as the bisection point of QQ' by construction 2.

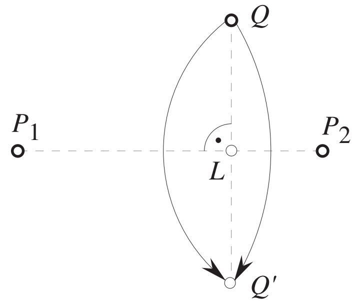

Figure 4

Let us now analyze the situation of two straight lines P_1P_2 and Q_1Q_2 intersecting in S (see Figure 5):

Let L be the footpoint of the perpendicular through Q_1 on P_1P_2 and N be the footpoint of the perpendicular through L on Q_1Q_2. Both L and N are obtained by construction 4. Hence we have the relation

$$(Q_1L)^2 = Q_1N \cdot Q_1S\,.$$

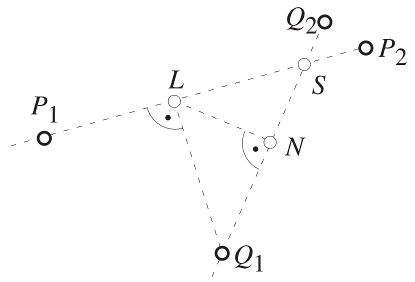

Figure 5

The idea is now to construct the length l of Q_1S since then we find S as intersection of Q_1Q_2 and a circle with center Q_1 and radius l (see construction 3) and we are through! In fact l is obtained as follows:

Construction 5

First we double Q_1L ($Q_1C = 2Q_1L$) according to Remark 1 (see Figure 6). Let K be an arbitrary (but large enough) circle through Q_1 and C and let D be a point of K with $LD = Q_1N$. Further let E the intersection of LD and K (see construction 1). Then LE has length l since we have $(Q_1L)^2 = Q_1L \cdot LC = LD \cdot LE = Q_1N \cdot LE$ by Euler's theorem on intersecting chords in a circle.

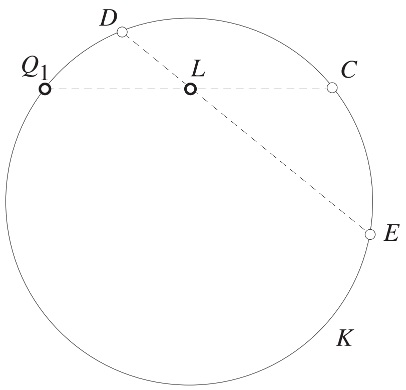

Figure 6

Bibliography

[1] Mascheroni Lorenzo: "Geometrie du compas", Coubron : Monom, 1980

[2] Mohr Georg: "Euclides danicus", Amsterdam : Van Velsen, 1672 (Kobenhavn: 1928.)

[3] Honsberger, R.: "Ingenuity in mathematics", Washington : Math. Association of America, 1970, ib.7 (New mathematical library, 23)

[4] Dörrie, H.: "100 famous problems in mathematics", München : Oldenbourg, 1960

[5] Kostovskii A. N.: "Geometrical Constructions Using Compasses Only", Blaisdell Publication Company, 1961

[6] Eves H.: "A Survey of Geometry", Allyn and Bacon, 1972

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